Model Order Reduction for Nonlinear Radiative Transfer Based on Moment Equations and Data-Driven Approximations of the Eddington Tensor

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Outline

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- Fundamental approach and basic ideas
  - Model order reduction for Boltzmann transport problems
  - Thermal radiative transfer (TRT) problems
  - Nonlinear projective approach
- Derivation of multilevel low-order Quasidiffusion (Variable Eddington Factor) equations for the Boltzmann transport equation (BTE)
- Formulation of data-driven ROMs for nonlinear TRT problems with POD and DMD of the Eddington tensor
- Numerical results for multigroup TRT problems in 2D Cartesian geometry
- Discussion
- This is a joint work with Dmitriy Anistratov (NCSU)

Introduction

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- Radiative transfer is the physical process of energy transfer via the propagation, emission and absorption of photon radiation in the host medium
- Radiative transfer becomes the dominant mode of energy redistribution in materials at extreme temperatures, and is an essential piece of physics for many phenomena spanning several fields
    - plasma physics
    - high-energy density physics
    - astrophysics
    - atmospheric science
- These phenomena are described by complex multiphysical systems of differential equations (radiation hydrodynamics)
- The particle transport physics is modeled by the Boltzmann transport equation (BTE)

Challenges in Simulation of Particle Transport Problems

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- The numerical simulation of mutiphysical particle transport problems faces several challenges
  - High-dimensionality
  - Strong nonlinearity
  - Strong coupling of equations
    - Material opacities depend on state of matter
    - The state of matter is influenced by particle fluxes
  - Multi-scale characterization
  - Distinct characteristic behavior in different energy ranges
  - System of equations of different types
  - Integro-differential BTE
- The BTE is especially challenging to solve
  - Hyperbolic differential operator
  - The solution at any point depends on the solution everywhere in phase space due to the integral operator
  - The solution is high-dimensional

Model Order Reduction for the BTE

- The BTE largely influences the dimensionality of these problems
  - 3D geometry $\rightarrow$ 7 independent variables
- Reduced order models (ROMs) for the BTE are commonly employed to reduce the cost of multiphysics simulations involving radiative transfer
- Some ROMs have seen widespread adoption for their usefulness and simplicity
  - Diffusion-based models like flux limited diffusion (FLD)
  - P_1, $P_{1/3}$, P_3
  - Minerbo models
- However, the accuracy of these models is limited
- The development of *advanced* ROMs for the BTE that can achieve both computational efficiency and high accuracy is an active field of research
  - In recent times the bulk of this research effort has been focused on data-driven ROMs

Data-Driven Model Order Reduction for the BTE

- The most advanced methods for data-driven model order reduction were developed with fluid dynamics in mind
- More recently these methods have been the subject of a substantial research effort for developing ROMs for Boltzmann transport
- The majority of these ROMs focus on linear particle transport problems
- Many questions remain for nonlinear problems of high energy density physics (radiation hydrodynamics problems)
  - How to preserve and reproduce essential features of fundamental physics
  - Dealing with multiscale behavior

Fundamental Approach

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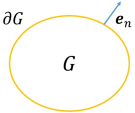
- Nonlinear projective approach
  - interpretation
    - nonlinear method of moments
    - multigrid approach
  - Known to give advantage in multiphysical, multiscale applications
- The BTE is coupled with a hierarchy of low-order moment equations
  - Each moment system is exactly closed through nonlinear functionals of the BTE solution (e.g. the Eddington tensor)
  - Moment equations are conservation equations for integral (low-dimensional) quantities
  - Moment equations account for different scales of the problem
  - Multiphysics equations are coupled to moment equations on the same dimensional scale
- Data-driven methods of approximation to estimate closures for low-order equations
  - proper orthogonal decomposition (POD)
  - dynamic mode decomposition (DMD)
  - Fundamental radiation physics conserved through moment equations

Nonlinear Thermal Radiative Transfer Problem

- Prototypes of these ROMs are designed for the fundamental thermal radiative transfer (TRT) problem
  - Supersonic radiation flow problem neglecting material motion, photon scattering, heat conduction and external sources
- The high-order multigroup Boltzmann transport equation (BTE)

$$\frac{1}{c}\frac{\partial I_g(\mathbf{r},\boldsymbol{\Omega},t)}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I_g(\mathbf{r},\boldsymbol{\Omega},t) + \varkappa_g(T) I_g(\mathbf{r},\boldsymbol{\Omega},t) = \varkappa_g(T) B_g(T)$$



$$\mathbf{r}\in G\,,\ \text{for all } \boldsymbol{\Omega},\quad g=1,\ldots,N_g,\quad t\geq t_0,$$
$$I_g|_{\mathbf{r}\in\partial G} = I_g^{in},\quad \boldsymbol{\Omega}\cdot\boldsymbol{e}_n<0,\quad t\geq t_0,$$
$$I_g|_{t=t_0} = I_g^{0},\quad \mathbf{r}\in G\,,\ \text{for all } \boldsymbol{\Omega},$$

  - $\mathbf{r}$ - spatial position, $\boldsymbol{\Omega}$ - direction of particle motion, g - photon frequency group, t - time, $I_g(\mathbf{r},\boldsymbol{\Omega},t)$ - specific intensity of photons in the group g
- The material energy balance (MEB) equation

$$\frac{\partial\varepsilon(T)}{\partial t} = \sum_{g=1}^{N_g}\int_{4\pi}\varkappa_g(T)\Big(I_g(\mathbf{r},\boldsymbol{\Omega},t) - B_g(T)\Big)d\boldsymbol{\Omega}\,,\quad T|_{t=t_0} = T^0(\mathbf{r}),\ \text{for } \mathbf{r}\in G.$$

Multigroup Quasidiffusion (VEF) Equations

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$$\boxed{\frac{1}{c}\frac{\partial I_g}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I_g + \varkappa_g(T) I_g = \varkappa_g(T) B_g(T)}$$

- Angular moments of intensities
  - $E_g((r), t) = \frac{1}{c}\int_{4\pi} I_g d\Omega$
  - $\boldsymbol{F}_g((r), t) = \int_{4\pi} \boldsymbol{\Omega} I_g d\Omega$
- Projection: $\int_{4\pi} \cdot d\Omega$ $\int_{4\pi} \boldsymbol{\Omega}\cdot d\Omega$

$$\frac{\partial E_g}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F}_g + c\varkappa_g(T) E_g = 4\pi\varkappa_g(T) B_g(T)\,, \tag{1}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{F}_g}{\partial t} + c\boldsymbol{\nabla}\cdot\left(\mathfrak{f}_g[I] E_g\right) + \varkappa_g(T)\boldsymbol{F}_g = 0 \tag{2}$$

- Exact closure through the Eddington tensor

$$\mathfrak{f}_g[I] = \frac{\int_{4\pi} \boldsymbol{\Omega}\otimes\boldsymbol{\Omega} I_g d\Omega}{\int_{4\pi} I_g d\Omega} \tag{3}$$

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Effective Grey Quasidiffusion (VEF) Equations

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$$\boxed{\begin{aligned}\frac{\partial E_g}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F}_g + c\varkappa_g(T)E_g &= 4\pi\varkappa_g(T)B_g(T),\\ \frac{1}{c}\frac{\partial \boldsymbol{F}_g}{\partial t} + c\boldsymbol{\nabla}\cdot\left(\mathfrak{f}_g[I]E_g\right) + \varkappa_g(T)\boldsymbol{F}_g &= 0\end{aligned}}$$

- Frequency-integrated angular moments of intensities
  - $E = \sum_{g=1}^{N_g} E_g, \quad \boldsymbol{F} = \sum_{g=1}^{N_g} \boldsymbol{F}_g$
- Projection: $\sum_{g=1}^{N_g}\cdot$

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} + c\bar{\varkappa}_E E = c\bar{\varkappa}_B a_R T^4 \tag{4}$$

$$\frac{1}{c}\frac{\partial \boldsymbol{F}}{\partial t} + c\boldsymbol{\nabla}\cdot(\bar{\mathfrak{f}}E) + \bar{\boldsymbol{K}}_R\boldsymbol{F} + \bar{\eta}E = 0 \tag{5}$$

- Exact closure through frequency-averaged grey factors

$$\bar{\varkappa}_E = \frac{\sum_{g=1}^{N_g}\varkappa_g E_g}{\sum_{g=1}^{N_g} E_g}, \quad \bar{\varkappa}_B = \frac{\sum_{g=1}^{N_g}\varkappa_g B_g}{\sum_{g=1}^{N_g} B_g}, \quad \bar{\varkappa}_{R,\alpha} = \frac{\sum_{g=1}^{N_g}\varkappa_g |F_{\alpha,g}|}{\sum_{g=1}^{N_g} |F_{\alpha,g}|}, \tag{6}$$

$$\boldsymbol{K}_R = \mathrm{diag}(\bar{\varkappa}_{R,x}, \bar{\varkappa}_{R,y}, \bar{\varkappa}_{R,z}), \quad \bar{\mathfrak{f}} = \frac{\sum_{g=1}^{N_g}\mathfrak{f}_g E_g}{\sum_{g=1}^{N_g} E_g}, \quad \bar{\eta} = \frac{\sum_{g=1}^{N_g}(\varkappa_g - \bar{\boldsymbol{K}}_R)\boldsymbol{F}_g}{\sum_{g=1}^{N_g} E_g}. \tag{7}$$

Effective Grey Quasidiffusion (VEF) Equations

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$$\boxed{\begin{aligned} \frac{\partial E_g}{\partial t} + \nabla \cdot \boldsymbol{F}_g + c\varkappa_g(T)E_g &= 4\pi\varkappa_g(T)B_g(T), \\ \frac{1}{c}\frac{\partial \boldsymbol{F}_g}{\partial t} + c\nabla \cdot \Big(\mathfrak{f}_g[I]E_g\Big) + \varkappa_g(T)\boldsymbol{F}_g &= 0 \end{aligned}}$$

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Multilevel Quasidiffusion Method

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- High-order Boltzmann transport equation for I_g

$$\frac{1}{c}\frac{\partial I_g}{\partial t} + \boldsymbol{\Omega}\cdot\boldsymbol{\nabla} I_g + \varkappa_g(T) I_g = \varkappa_g(T) B_g(T)$$

- Eddington tensor $I_g \Rightarrow \mathfrak{f}_g[I] = \frac{\int_{4\pi} \boldsymbol{\Omega}\otimes\boldsymbol{\Omega} I_g \, d\Omega}{\int_{4\pi} I_g \, d\Omega}$

- Multigroup quasidiffusion equations for E_g, $\boldsymbol{F}_g$

$$\frac{\partial E_g}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F}_g + c\varkappa_g(T)E_g = 4\pi\varkappa_g(T)B_g(T),$$

$$\frac{1}{c}\frac{\partial \boldsymbol{F}_g}{\partial t} + c\boldsymbol{\nabla}\cdot\left(\mathfrak{f}_g[I]E_g\right) + \varkappa_g(T)\boldsymbol{F}_g = 0$$

- Grey closures E_g, $\boldsymbol{F}_g \Rightarrow \bar{\varkappa}_E,\ \bar{\varkappa}_B,\ \bar{\mathfrak{f}},\ \bar{\boldsymbol{K}}_{\boldsymbol{R}},\ \bar{\boldsymbol{\eta}}$

- Effective grey problem for E, $\boldsymbol{F}$, T

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla}\cdot\boldsymbol{F} + c\bar{\varkappa}_E E = c\bar{\varkappa}_B a_R T^4$$

$$\frac{1}{c}\frac{\partial \boldsymbol{F}}{\partial t} + c\boldsymbol{\nabla}\cdot(\bar{\mathfrak{f}}E) + \bar{\boldsymbol{K}}_{\boldsymbol{R}}\boldsymbol{F} + \bar{\boldsymbol{\eta}}E = 0$$

$$\frac{\partial \varepsilon(T)}{\partial t} = c\left(\bar{\varkappa}_E E - \bar{\varkappa}_B a_R T^4\right)$$

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Reduced-Order Model for TRT

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- Data-driven approximator

$$\tilde{\mathfrak{f}}_g = \mathcal{G}[\mathfrak{f}^*], \quad \mathfrak{f}^* \text{ known}$$

- Multigroup quasidiffusion equations for E_g, $\boldsymbol{F}_g$

$$\frac{\partial E_g}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F}_g + c\varkappa_g(T)E_g = 4\pi\varkappa_g(T)B_g(T),$$

$$\frac{1}{c}\frac{\partial \boldsymbol{F}_g}{\partial t} + c\boldsymbol{\nabla} \cdot \left(\tilde{\mathfrak{f}}_g E_g\right) + \varkappa_g(T)\boldsymbol{F}_g = 0$$

- Grey closures E_g, $\boldsymbol{F}_g \Rightarrow \bar{\varkappa}_E$, $\bar{\varkappa}_B$, $\bar{\mathfrak{f}}$, $\bar{\boldsymbol{K}}_{\boldsymbol{R}}$, $\bar{\boldsymbol{\eta}}$

- Effective grey problem for E, $\boldsymbol{F}$, T

$$\frac{\partial E}{\partial t} + \boldsymbol{\nabla} \cdot \boldsymbol{F} + c\bar{\varkappa}_E E = c\bar{\varkappa}_B a_R T^4$$

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Approximation of Eddington Tensor

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- Projection, compression of data
  - Given: full-order solution to some TRT problem for N_t time steps, N_g frequency groups, N_r spatial cells
  - Vectors of each Eddington Tensor component at time t^n

$$\mathbf{f}^n_{\alpha\beta} \in \mathbb{R}^{N_r N_g}, \quad \alpha, \beta = x, y, z$$

  - Construction of snapshot matrices

$$\mathbf{A}^{\mathfrak{f}_{\alpha\beta}} = [\mathbf{f}^1_{\alpha\beta}\ \mathbf{f}^2_{\alpha\beta}\ \dots\ \mathbf{f}^{N_t}_{\alpha\beta}] \in \mathbb{R}^{N_r N_g \times N_t}$$

  - Define projection operator $\mathcal{G}^k$ that will project a given matrix $\mathbf{A}$ onto a rank-k subspace

$$\mathcal{A}^{\mathfrak{f}_{\alpha\beta}}_k = \mathcal{G}^k \mathbf{A}^{\mathfrak{f}_{\alpha\beta}}, \quad k \leq \text{rank}(\mathbf{A}^{\mathfrak{f}_{\alpha\beta}})$$

  - $\mathcal{A}^{\mathfrak{f}_{\alpha\beta}}_k$ is constructed of k sets of various vectors and factors
- Approximation of data from rank k representation
  - A map $\mathcal{M}^n$ is defined by the same method used to define $\mathcal{G}^k$

$$\mathbf{f}^n_{\alpha\beta} \approx \mathcal{M}^n \mathcal{A}^{\mathfrak{f}_{\alpha\beta}}_k$$

POD of the Eddington Tensor

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- *Centered* data matrix

$$\hat{\mathbf{A}}^{\mathfrak{f}_{\alpha\beta}} = [\hat{\mathbf{f}}^1_{\alpha\beta}, \dots, \hat{\mathbf{f}}^{N_t}_{\alpha\beta}], \qquad \hat{\mathbf{f}}^n_{\alpha\beta} = \mathbf{f}^n_{\alpha\beta} - \bar{\mathbf{f}}_{\alpha\beta}, \qquad \bar{\mathbf{f}}_{\alpha\beta} = \frac{1}{N_t}\sum_{n=1}^{N_t} \mathbf{f}^n_{\alpha\beta}$$

- A thin singular value decomposition (SVD) represents the matrix in the form

$$\hat{\mathbf{A}}^{\mathfrak{f}_{\alpha\beta}} = \mathbf{U}\mathbf{S}\mathbf{V}^T,$$

$\mathbf{U} \in \mathbb{R}^{N_r N_g, d}$ holds the left singular vectors $\{\boldsymbol{u}_\ell\}_{\ell=1}^d$ in its columns
$\mathbf{V} \in \mathbb{R}^{N_t, d}$ holds the right singular vectors $\{\boldsymbol{v}_\ell\}_{\ell=1}^d$ in its columns
$\mathbf{S} \in \mathbb{R}^{d,d}$ holds the singular values $\{\sigma_\ell\}_{\ell=1}^d$ along its diagonal in decreasing order,

$$d = \mathrm{rank}(\hat{\mathbf{A}}^{\mathfrak{f}_{\alpha\beta}}) = \min(N_r N_g, N_t)$$

- The rank-k POD representation of $\mathfrak{f}_{\alpha\beta}$

$$\mathcal{A}_k^{\mathfrak{f}_{\alpha\beta}} = \bar{\mathbf{f}}_{\alpha\beta} \cup \{\sigma_\ell, \boldsymbol{u}_\ell, \boldsymbol{v}_\ell\}_{\ell=1}^k$$

$$\boxed{\tilde{\mathbf{f}}^n_{\alpha\beta} \leftarrow \bar{\mathbf{f}}_{\alpha\beta} + \sum_{\ell=1}^{k} \sigma_\ell (\boldsymbol{v}_\ell)_n \boldsymbol{u}_\ell = \mathcal{M}^n \mathcal{A}_k^{\mathfrak{f}_{\alpha\beta}}}$$

DMD of the Eddington Tensor

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- The DMD constructs the best-fit linear operator $\mathbf{B}$ to the time-dependent data $\{\mathbf{f}^n_{\alpha\beta}\}_{n=1}^{N_t}$, generating the dynamic system:

$$\frac{d\tilde{\mathbf{f}}_{\alpha\beta}(t)}{dt} = \mathbf{B}\tilde{\mathbf{f}}_{\alpha\beta}(t), \quad \Rightarrow \quad \tilde{\mathbf{f}}_{\alpha\beta}(t) = \sum_{\ell=1}^{k} \beta_\ell \boldsymbol{\varphi}_\ell e^{\omega_\ell t},$$

  $\omega_\ell, \boldsymbol{\varphi}_\ell$ are the eigenvalues and associated eigenfunctions of $\mathbf{B}$

- Form two data matrices for $t^n = t^{n-1} + \Delta t$ with constant time step

$$\mathbf{X}_{\alpha\beta} = [\mathbf{f}^1_{\alpha\beta}, \dots, \mathbf{f}^{N_t-1}_{\alpha\beta}], \quad \widehat{\mathbf{X}}_{\alpha\beta} = [\mathbf{f}^2_{\alpha\beta}, \dots, \mathbf{f}^{N_t}_{\alpha\beta}] \quad \mathbf{f}^n_{\alpha\beta} \in \mathbb{R}^{N_r N_g}$$

- $\tilde{\mathbf{B}} = \hat{\mathbf{X}}\mathbf{X}^+$ is the closest approximation to $\mathbf{B}$ in the Frobenius norm. Here $+$ signifies the Moore-Penrose pseudo inverse.

- The DMD representation

$$\boxed{\mathcal{A}_k^{\mathbf{f}_{\alpha\beta}} = \{\tilde{\beta}_\ell, \tilde{\omega}_\ell, \tilde{\boldsymbol{\varphi}}_\ell\}_{\ell=1}^{k}, \quad \tilde{\mathbf{f}}_{\alpha\beta}(t^n) = \sum_{\ell=1}^{k} \tilde{\beta}_\ell \tilde{\boldsymbol{\varphi}}_\ell e^{\tilde{\omega}_\ell t^n}, \quad \tilde{\omega}_\ell = \Delta t^{-1} \ln(\gamma_\ell),}$$

  where $\gamma_\ell, \tilde{\boldsymbol{\varphi}}_\ell$ are the eigenvalues and associated eigenfunctions of $\tilde{\mathbf{B}}$

- Equilibrium-subtracted DMD (DMD-E)
  - Construction of the linear operator $\mathbf{B}$ that fits the equilibrium-subtracted data
  - The time-dependent TRT problem tends to steady-state as $t \to \infty$.

Determination of Rank

- For each data matrix $\mathbf{A}$, the rank k must be calculated to store $\mathcal{A}_k$
- We consider the rank-k truncated SVD

$$\mathbf{A} \approx \mathbf{A}_k = \mathbf{U}_k \mathbf{S}_k \mathbf{V}_k^T$$

$$\mathbf{U}_k = [\mathbf{u}_1 \dots \mathbf{u}_k], \quad \mathbf{S}_k = \text{diag}(\sigma_1, \dots, \sigma_k), \quad \mathbf{V}_k = [\mathbf{v}_1 \dots \mathbf{v}_k]$$

- The error in Frobenius norm

$$\|\mathbf{A} - \mathbf{A}_k\|_F^2 = \sum_{\ell=k+1}^{d} \sigma_\ell^2$$

- We calculate k by choosing the relative Frobenius norm error ξ_{rel}

$$\boxed{\xi_{\text{rel}}^2 = \frac{\sum_{\ell=k+1}^{d} \sigma_\ell^2}{\sum_{\ell=1}^{d} \sigma_\ell^2} = \frac{\|\mathbf{A} - \mathbf{A}_k\|_F^2}{\|\mathbf{A}\|_F^2}}$$

- k increases as ξ_{rel} decreases
  - Increasing k increases cost in calculating approximate closures
  - Accuracy is also expected to increase with k

- We do not have theory to predict how the models will perform
- Numerical testing is used to demonstrate the following
  - The models converge to the reference solution as $k \to d$
  - How the ROM errors converge as $k \to d$
  - How well the *low-rank* ROMs capture fundamental physics in the solution compared to the full-order TRT model on a given phase-space and time

Numerical Test Problem

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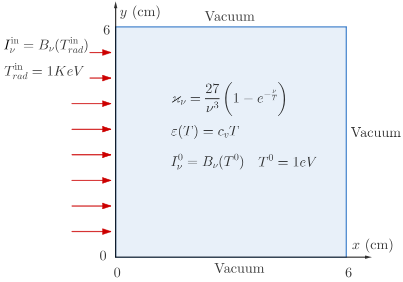

- We test this method on a 2D extension of the Fleck-Cummings (F-C) test
- Fully-implicit time integration
- BTE discretized with simple corner balance scheme
- All low-order equations discretized with 2nd order finite volumes
- Grid:
  - 20x20 spatial cells, 17 groups, 144 discrete directions
  - 300 time steps $\Delta t = .02$ns for $0 \leq t \leq 6$ns
- Degrees of freedom: 1.175×10^9
- ROM degrees of freedom: 1.728×10^7

Numerical Results (POD)

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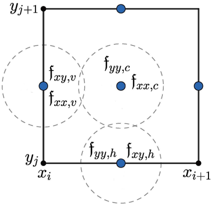
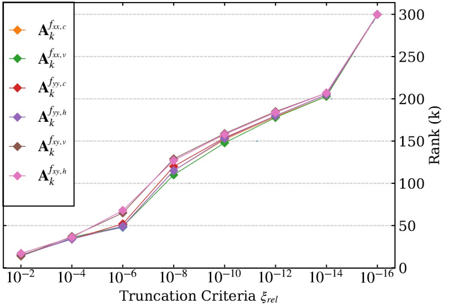
- Databases are formed from the full-order model solution
- Ranks of approximation and memory requirements for the POD with varying ξ_{rel}





Ranks k vs parameter ξ_{rel}

- I : $N_x N_y N_\Omega N_g \times N_t = 1.18 \times 10^9$
- $\boldsymbol{A}^{\mathfrak{f}_{xx,c}}$: $N_x N_y N_g \times N_t = 2.04 \times 10^6$
- $\mathcal{A}_k^{\mathfrak{f}_{xx,c}}$: $k(1 + N_t) + (k+1) N_x N_y N_g$
- $\xi_{\text{rel}} = 10^{-2} \rightarrow k = 15 : \; 1.13 \times 10^5$
- $\xi_{\text{rel}} = 10^{-4} \rightarrow k = 34 : \; 2.48 \times 10^5$

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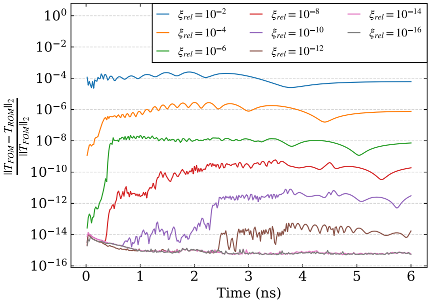
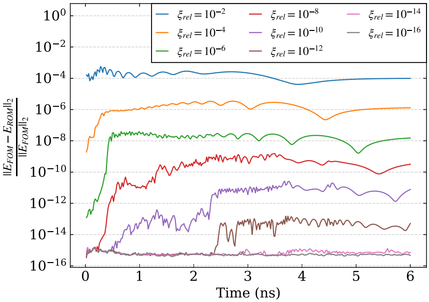
- Relative errors in the 2-norm using the POD compared to the full-order solution
- Errors plotted vs time, each curve corresponds to a value for ξ_{rel}



Material Temperature



Radiation Energy Density

Numerical Results (DMD)

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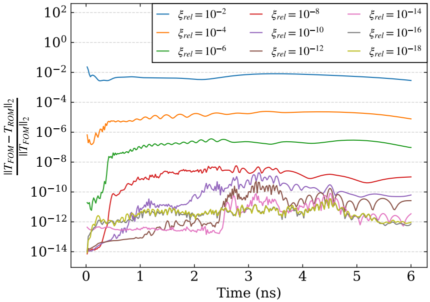
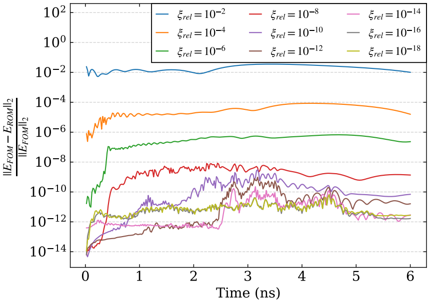
- Relative errors in the 2-norm using the DMD compared to the full-order solution
- Errors plotted vs time, each curve corresponds to a value for ξ_{rel}



Material Temperature



Radiation Energy Density

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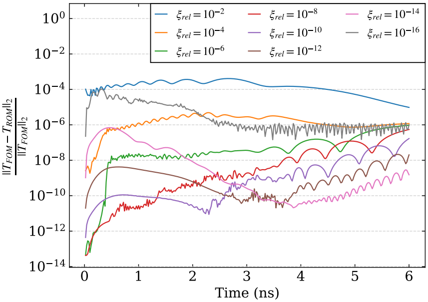
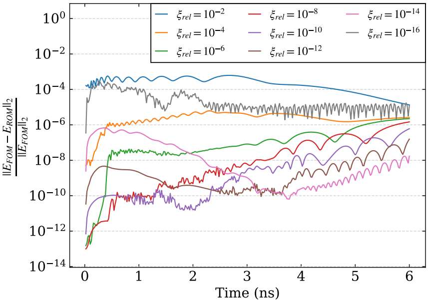
- Relative errors in the 2-norm using the DMD-E compared to the full-order solution
- Errors plotted vs time, each curve corresponds to a value for ξ_{rel}



Material Temperature



Radiation Energy Density

Convergence of the ROM with Rank (POD)

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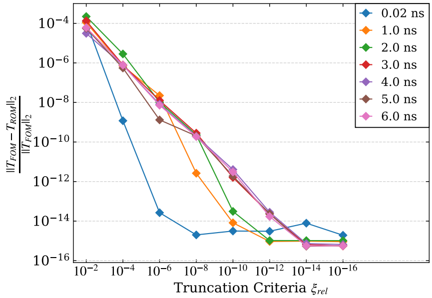
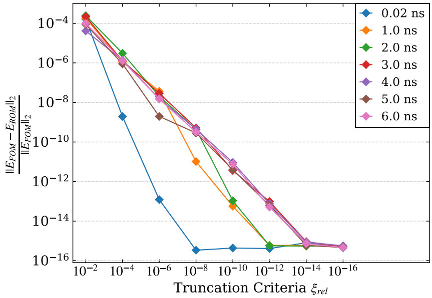
- Relative errors in the 2-norm using the POD compared to the full-order solution
- Errors plotted vs ξ_{rel}, each curve corresponds to an instant of time



Material Temperature



Radiation Energy Density

Convergence of the ROM with Rank (DMD)

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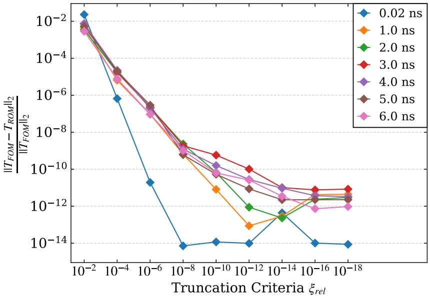
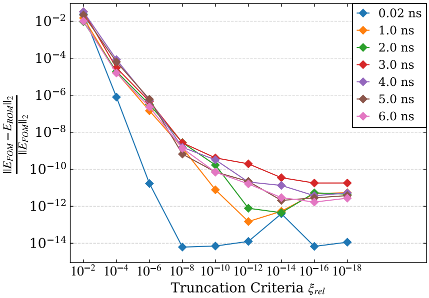
- Relative errors in the 2-norm using the DMD compared to the full-order solution
- Errors plotted vs ξ_{rel}, each curve corresponds to an instant of time



Material Temperature



Radiation Energy Density

Convergence of the ROM with Rank (DMD-E)

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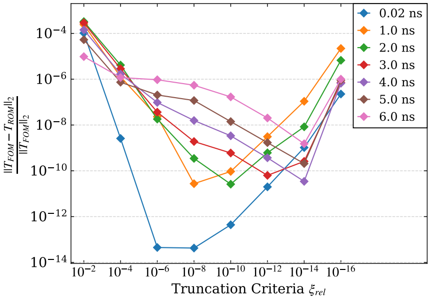
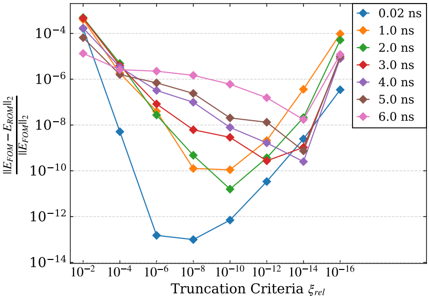
- Relative errors in the 2-norm using the DMD-E compared to the full-order solution
- Errors plotted vs ξ_{rel}, each curve corresponds to an instant of time



Material Temperature



Radiation Energy Density

Conservation of Physics: Breakout Time

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- To demonstrate how the low-rank ROMs conserve fundamental physics in the TRT solution, we consider breakout time
- Breakout time is an important measurement in high-temperature laser-driven radiation shock experiments
- For the F-C test we can calculate how well the ROMs capture certain quantities on the right boundary as time elapses
- Total radiation energy density ($\bar{E}_R$) and material temperature ($\bar{T}_R$) averaged over the right boundary of the spatial domain vs time

$$\bar{T}_R = \frac{1}{L_R}\int_0^{L_R} T(x_R, y)\ dy\,, \quad \bar{E}_R = \frac{1}{L_R}\int_0^{L_R} E(x_R, y)\ dy$$

- We also consider the spectrum of radiation present at the right boundary

$$\bar{E}_{R,g} = \frac{1}{L_R}\int_0^{L_R} E_g(x_R, y)\ dy$$

Breakout Time: FOM

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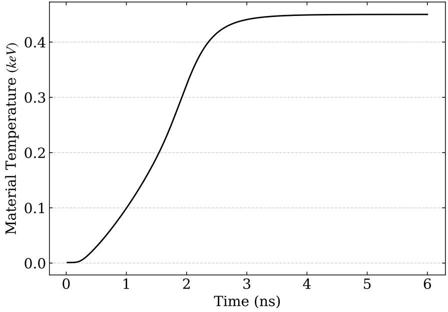
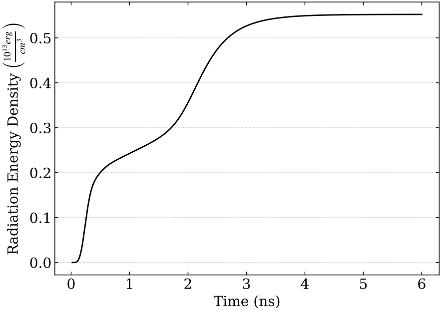
- Total radiation energy density ($\bar{E}_R$) and material temperature ($\bar{T}_R$) averaged over the right boundary of the spatial domain vs time

$$\bar{T}_R = \frac{1}{L_R}\int_0^{L_R} T(x_R, y)\, dy\,, \quad \bar{E}_R = \frac{1}{L_R}\int_0^{L_R} E(x_R, y)\, dy$$



$\bar{T}_R$

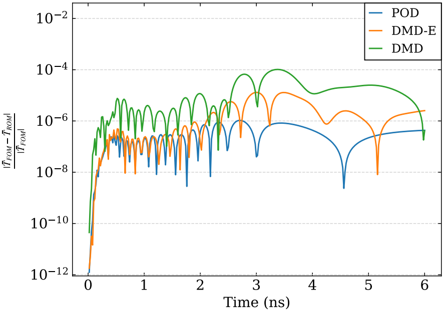


$\bar{E}_R$

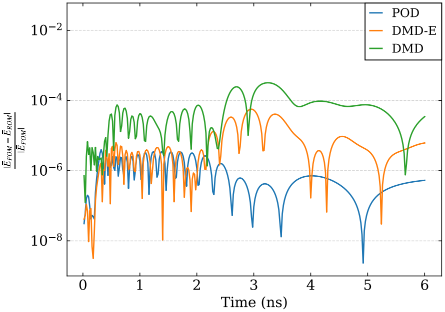
Breakout Time: Relative Error for the ROMs

- $\xi_{\text{rel}} = 10^{-4}$

$$\bar{T}_R = \frac{1}{L_R}\int_0^{L_R} T(x_R, y)\ dy\ , \quad \bar{E}_R = \frac{1}{L_R}\int_0^{L_R} E(x_R, y)\ dy$$



$\bar{T}_R$



$\bar{E}_R$

Breakout Time: Relative Error for the DDET ROMs

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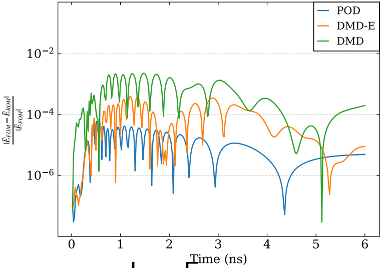
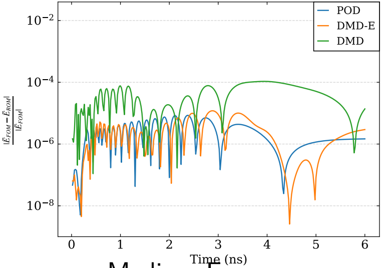
- $\xi_{\text{rel}} = 10^{-4}$

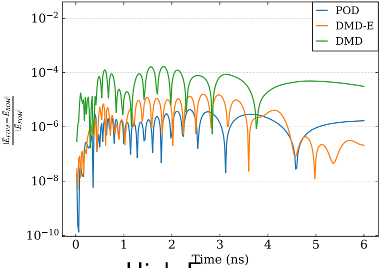
$$\bar{E}_{R,g} = \frac{1}{L_R}\int_0^{L_R} E_g(x_R, y)\ dy$$



Low Energy



Medium Energy



High Energy

Conclusion

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- Main elements of the developed methodology
  - nonlinear projection of the BTE and formulation of a hierarchy of low-order moment equations
  - data-driven techniques to approximate closures
- All models require data on the BTE solution
- The proposed methods have been shown to perform well on highly nonlinear thermal radiative transfer problems
- Any multiphysical equations of interest can be coupled with the effective grey QD equations
- Use of the low-order moment equations enforces conservation of fundamental physics in radiative transfer component
- The proposed approach for development of ROMs can be applied to a wide class of multiphysical high-energy density problems, such as radiative hydrodynamics problems.
- Future research items
  - Optimal sampling techniques for generation of data to inform and train ROMs for TRT
  - Investiagion of more complex data-driven methods for approximation
  - Parametrized ROMs for TRT

Publications

- J. Coale and D. Anistratov, A reduced-order model for thermal radiative transfer problems based on multilevel quasidiffusion method, *Proc. of Int. Conf. on Mathematics and Computational Methods Applied to Nuclear Science and Engineering M&C 2019*, Portland, OR.
- J. Coale and D. Anistratov, Data-driven grey reduced-order model for thermal radiative transfer problems based on low-order quasidiffusion equations and proper orthogonal decomposition, *Transactions of American Nuclear Society*, **121** (2019)
- J. Coale and D. Anistratov, Reduced order models for nonlinear radiative transfer based on moment equations and POD/DMD of Eddington tensor, *arXiv*: 2107.09174 (2021)
- J. Coale and D. Anistratov, Reduced-order models for thermal radiative transfer based on POD-Galerkin method and low-order quasidiffusion equations, *Proc. of Int. Conf. on Mathematics and Computational Methods Applied to Nuclear Science and Engineering M&C 2021*, Raleigh, NC.

Questions?